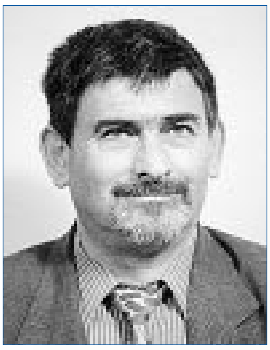

STATISTICAL CHECKING

Mathematical control of the randomness of gambling devices

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Abstract

In some cases, mathematical statistical checking is the only way to adequately test gambling devices and the two most frequently used methods (i.e. the 3σ method and the χ^2 method) are discussed in this paper. Because the χ^2 method cannot be used in its classical form for testing lottery-type games, it was necessary to develop it so that it could also be applicable for this purpose. For both roulette and lottery-type games, the second and the third members in the series of the distribution function of the variable χ^2 are also determined in order to form an improved test method.

1 Introduction

In Hungary, as in some other European countries, the technical control of gambling devices is legally the task of the national metrological institute; mathematical statistical checking of the random (and therefore fair) operation of such devices also falls under the institute's responsibility. In some cases huge prizes are drawn by simple devices made of wood, whose structure cannot be tested by classical metrological means; mathematical checking of these devices is therefore the only possible official control method. Considering the fact that in this field written standards are not available, it is vital that the authority's decision as to whether a device is accepted or rejected be well-founded and indisputable.

This paper shows the mathematical bases of the two most important statistical methods mentioned above, which are most frequently used for type approval and verification. These mathematical methods can be used not only for gambling devices but also for other purposes, for example to check a hypothesis about a probability distribution or to check a computer's random number generator.

In some cases during the author's research it was necessary to further expand on existing methods, or even to elaborate new ones; these can be found in paragraphs 4, 5 and 6 below.

2 The 3σ test

The method is given for the case of roulette wheels but can, of course, be used in other fields too.

Roulette wheels used in Hungary have numbers 1 to 36 plus a single zero, so the number of possible spin outcomes equals 37, usually designated by v. One of the steps in testing the randomness is to check the hypothesis that the probability of spinning any number out of the 37 is the same:

$$p_j = \frac{1}{v} \quad (j = 0, 1, \ldots, 36).$$

When a roulette wheel is submitted for testing the client must provide a list of the last N spins, in sequential order. Such lists used to be written by hand, though nowadays they are made electronically using a data collector connected to the corresponding roulette table electronic display device. The i^{th} number on the list, i.e. one of the numbers 0 to 36, is designated by x_i ($i = 1, 2, \ldots, N$) and N is prescribed not to be less than $100v = 3700$ in order to ensure that all 37 numbers occur at least 100 times *on average*. The prescribed number $N = 100v$ can really be judged as being a compromise: sometimes clients consider it as being too big, but to ascertain the smaller deviations of a roulette wheel more data would be necessary.

On the basis of the list the frequencies k_j of the occurrence of the numbers j ($j = 0, 1, \ldots, 36$) are determined, where the frequency k_j shows how many times the number j occurred among the values x_i.

In the case of a roulette wheel operating regularly, i.e. randomly, every frequency k_j is a random variable following a Bernoulli distribution with an expectation:

$$\mu_j \equiv \langle k_j \rangle = Np_j = \frac{N}{\nu}$$

and with a theoretical standard deviation:

$$\sigma_j \equiv +\sqrt{\langle (k_j - \mu_j)^2 \rangle} = +\sqrt{Np_j(1-p_j)} = +\sqrt{\frac{N}{\nu}\left(1-\frac{1}{\nu}\right)}$$

So for every frequency k_j the condition:

$$\boxed{\mu_j - 3\sigma_j \le k_j \le \mu_j + 3\sigma_j}$$

has to be fulfilled with a high probability, the value of which is approximately 99.73 %. The factor 3 that precedes σ_j gave this test its name, which - according to the author's experience - can be judged as being optimal. The choice of a smaller factor would result in a higher risk of the *first order error,* when roulette wheels operating correctly would be rejected more often, since the data would be outside the interval more often. The choice of a higher factor would increase the risk of the *second order error*, when unacceptable roulette wheels would be accepted.

The above method is used by most roulette wheel manufacturers to check their product, with the modification that the frequencies of odd/even, red/black, etc. numbers are also examined. However, this test only controls the "uniformity" of the frequencies of the numbers spun, and says nothing about their randomness. A roulette wheel spinning consecutively increasing numbers (e.g. 15, 16, 17 or 35, 36, 0, etc.) would pass the above test properly; every frequency k_j would fall *in the center* of the prescribed interval, despite the fact that the "decisions" of the given roulette wheel would be *extremely predictable* and not random at all. The operation of a roulette wheel is deemed to be random if no regularity can be found in adjacent numbers. *Hence the randomness of the sequence of the numbers spun must also be checked.*

For this purpose one can employ several methods, for instance the correlation method. According to the author's experience one of the simplest, most efficient and most demonstrative methods is to take the difference between the numbers in sequence:

$$y_{i,1} \equiv x_{i+1} - x_i \quad (i = 1, 2, ..., N-1).$$

Since the previous differences can be negative, the next non-negative quantities $z_{i,1}$ are formed:

$$z_{i,1} \equiv \begin{cases} y_{i,1} & \text{if } y_{i,1} \ge 0 \\ y_{i,1} + \nu & \text{if } y_{i,1} < 0 \end{cases} \qquad (1)$$

The so-called *modulated differences* $z_{i,1}$ can have values between 0 and 36 too, and if the roulette wheel operates regularly the distribution of the frequencies k_j, calculated on the basis of the values $z_{i,1}$, must also obviously be approximately uniform, similarly to the frequencies k_j calculated from the original data in the list. Hence for the frequencies k_j calculated on the basis of the values $z_{i,1}$, the foregoing 3σ test is also performed, taking $(N-1)$ instead of N. A wheel spinning numbers incrementally would fail this test; the frequency k_1 belonging to the "channel" number 1 (i.e. belonging to $z_{i,1} = 1$) would be much bigger than the upper limit of the given interval and all other frequencies k_j would equal zero.

Considering that the 3σ test is used not only for roulette wheels, but to control computer programs drawing prizes, for example, the above test is performed not only for the differences $z_{i,1}$, calculated from the adjacent numbers on the original list, but for the modulated differences $z_{i,k}$, that are calculated from the k^{th} neighbors. The $z_{i,k}$ modulated differences are defined as follows:

$$y_{i,k} \equiv x_{i+k} - x_i; \text{ here } k = 1, 2, 3, ... \text{ and } i = 1, 2, ..., N-k.$$

$$z_{i,k} \equiv \begin{cases} y_{i,k} & \text{if } y_{i,k} \ge 0 \\ y_{i,k} + \nu & \text{if } y_{i,k} < 0 \end{cases}$$

Using this method the gross errors, the short periodicity of the random generator of the computer or of the drawing program can be found out.

Performing the test for the 4th neighbors too, there are 5 × 37 = 185 frequencies on the data page of a roulette wheel. Since "only" P = 0.9973 probability belongs to 3σ, among these data sometimes (but regularly) some frequencies do not fulfil the 3σ condition, however the roulette wheel under control operates well. In these cases the decision to accept or reject it is partly subjective; the quantity, the place (the column and the line in the page) and the magnitude of the "overstepping" are examined.

The great advantage of the 3σ test is that it can generally be used for most gambling games such as slot machines or for lottery-type games, and in the case of rejection it gives an indication as to the possible reason for the deviation. However the 3σ test does have a disadvantage: unambiguous "mechanical" decisions cannot be made in the case of overstepping, i.e. when one or more frequencies are outside the given interval. Partly for this reason and also to render the decision better founded, as an addition to the 3σ method sometimes the χ^2 test is also performed on the same data. In other cases only the χ^2 method is used.

3 The χ^2 test for roulette-type games

Games are considered as being of the roulette-type where, at least in principle, it is possible to witness the same phenomenon of successive incremental spins described above. Roulette wheels (and most slot machines) belong to this category for instance. In the case of roulette wheels, repetitions of numbers must occur regularly. From the point of view of mathematics, roulette-type games differ from lottery-type games to a significant degree: in the latter, the numbers drawn during one game are always different. Such games are, for instance, the 90/5 lottery, where five different numbers are always drawn out of 90, the 45/6 lottery, the 80/20 keno, and all types of bingo games.

The χ^2 test is based on the very important statistical theorem described below, which is widely used for checking hypotheses on probability distributions.

Let the random events $A_1, A_2, \ldots, A_j, \ldots, A_v$ mutually exclude each other and let them constitute a complete system of events. In this case for the probabilities $p_1, p_2, \ldots, p_j, \ldots, p_v$ of the events the condition:

$$p_1 + p_2 + \ldots + p_j + \ldots + p_v = 1$$

is true, i.e. during one experiment one (and only one) event out of v different possible events will occur. Let k_j designate the frequency of the event A_j occurring out of N experiments where, of course, $k_j = 0, 1, 2, \ldots, N$ and

$$\sum_{j=1}^{v} k_j = N$$

(it is important to note that the greatest possible value of each frequency k_j equals N, which is the total of all the numbers spun in the case of roulette wheels). The set of frequencies k_j follows a v-variable Bernoulli distribution:

$$P(k_1, k_2, \ldots, k_v) = \frac{N!}{k_1!\, k_2! \ldots, k_v!}\; p_1^{k_1} p_2^{k_2} \ldots p_v^{k_v}$$

and according to the theorem the distribution limit of the next random variable constituted from the frequencies k_j:

$$\chi_R^2 \equiv \sum_{j=1}^{v} \frac{(k_j - Np_j)^2}{Np_j} \qquad (2)$$

is a χ^2 distribution with $r = v - 1$ degrees of freedom, if $N \to \infty$. Using the formulae, if $N \to \infty$:

$$Pr(\chi_R^2 < x) \equiv \phi_{\chi_R^2}(x) = F_{v-1}(x) \qquad (3)$$

Here the index R of the variable χ^2 indicates the roulette-type game, and $F_{v-1}(x)$ designates the distribution function of the χ^2 distribution with $r = v - 1$ degrees of freedom. Developed over the course of time, the fact that a random variable and a probability distribution are traditionally designated by the same symbol χ^2 may be inconvenient, though the random variable χ_R^2 appearing in (2) above is of an χ^2 distribution only in the case when $N \to \infty$, and not in general.

As a reminder, the density function f_r, the distribution function F_r and the first two moments of the χ^2 distribution with r degrees of freedom are given, which have the greatest importance during practical use:

$$f_r(x) = \frac{1}{\Gamma\left(\frac{r}{2}\right)} \frac{1}{x} \left(\frac{x}{2}\right)^{\frac{r}{2}} e^{-\frac{x}{2}}$$

$$F_r(x) \equiv \int_0^x f_r(y)dy = \frac{1}{\Gamma\left(\frac{r}{2}\right)} \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(k + \frac{r}{2}\right)k!} \left(\frac{x}{2}\right)^{k+\frac{r}{2}}$$

Here the Γ function is traditionally defined as:

$$\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy$$

If x is an integer, then $\Gamma(x) = (x - 1)!$ The expectation of the χ^2 distribution with r degrees of freedom

$$\mu \equiv \int_0^\infty x f_r(x)dx = r$$, and its variance

$$\sigma^2 \equiv \int_0^\infty (x - \mu)^2 f_r(x)dx = 2r$$

For practical purposes it may also be important that the χ^2 distribution with r degrees of freedom can be well approximated by a normal distribution, whose expectation equals r and whose standard deviation equals $\sqrt{2r}$, if the degree of freedom r is big enough.

When performing the mathematical test on roulette wheels, for instance, the χ^2 method can be used as follows:

a) on the basis of the data in the list sent in by the client, the frequencies k_j are determined;

b) supposing that the $p_j = \frac{1}{v} = \frac{1}{37}$ condition is true, i.e. the roulette wheel under control operates regularly, each number is spun with the same probability, on the basis of formula (2), the value of the variable χ_R^2 is calculated;

c) for an appropriate probability P of acceptance (the value of which usually equals 0.9973 belonging to the 3σ), supposing that the number N of the data is big enough for formula (3) to be used, the critical value χ^2_{crit} is determined from the condition

$$P = \phi_{\chi^2_R}(x = \chi^2_{crit});$$

d) the values of the variable χ^2_R and the critical values χ^2_{crit} are also determined for the modulated differences $z_{i,k}$, if necessary.

From a mathematical point of view the roulette wheel can be accepted if the condition(s):

$$\boxed{\chi^2_R \le \chi^2_{crit}}$$

are fulfilled for the base data and for the modulated differences $z_{i,k}$ as well.

The great benefit of the χ^2 test is that it always gives the possibility to make an unambiguous decision, but its disadvantage is that it does not give any information about the possible reasons in the case of rejection.

4 Improvement of the χ^2 test for roulette-type games

The χ^2 test is used not only for checking roulette wheels, where the condition $N >> 1$, necessary for the use of formula (3), is fulfilled in practice, but its application would sometimes be useful when the number N of the experiments is not big enough or if it cannot be ascertained whether it is big enough. In these cases the decision of the authority based on mathematical tests would not be well enough founded and sufficiently indisputable. For this reason the distribution function $\phi_{\chi^2_R}(x)$ of the random variable

$$\chi^2_R \equiv \sum_{j=1}^{v} \frac{(k_j - Np_j)^2}{Np_j}$$

defined by formula (2), was examined in detail, and it was established that the relation $\phi_{\chi^2_R}(x) = F_{v-1}(x)$, that is true in the case of $N \to \infty$, is the first member, independent of N, of a series according to the powers of

$$\frac{1}{\sqrt{N}}$$

In order to make the distribution function more exact the second and the third members of the series were determined too. Since the second member, which is proportional to

$$\frac{1}{\sqrt{N}}$$

equals zero, the distribution function of the variable χ^2_R is:

$$\phi_{\chi^2_R}(x) = F_{v-1}(x) + \frac{2x}{v-1} f_{v-1}(x) \left[-A - B + \frac{x(A+2B) - \frac{Bx^2}{v+3}}{v+1} \right] + O\left(\frac{1}{N^{3/2}}\right)$$

where:

$$A = \frac{v^2 + 2v - 2 - \sum_{j=1}^{v} \frac{1}{p_j}}{8N}$$

$$B = \frac{5\sum_{j=1}^{v} \frac{1}{p_j} - 3v^2 - 6v + 4}{24N}$$

and $f_{v-1}(x)$ and $F_{v-1}(x)$ are the density function and distribution function of the χ^2 distribution with $v - 1$ degrees of freedom, respectively.

The previous formulae contain the expression $\sum_{j=1}^{v} \frac{1}{p_j}$ which is small for a fixed value of v when all the probabilities p_j are almost equal, i.e. when the probability distribution of the events A_j is approximately uniform. This fact confirms the rule, as can be seen from the literature, that prescribes almost uniform distribution for the successful use of the first approximation, i.e. of the classical χ^2 test. According to that rule it is also necessary for every event A_j to occur at least 10 times. In the case of slot machines neither the first nor the second conditions can be fulfilled: the probability distribution of the different winning combinations occurring differs from the uniform distribution to a significant degree, and it cannot be ensured either during all the tests that the very infrequent "jackpot" will occur 10 times at least.

Hence considering the second and the third member in the series makes it possible to *prove* the correctness of using the first approximation or to make well-founded decisions even in the cases when the conditions necessary for the use of the first approximation cannot be fulfilled.

5 The χ^2 test for lottery-type games

If out of the numbers 1, 2, ..., j, ..., v during one draw n different numbers are drawn, for instance in the case of a 90/5 lottery $n = 5$ different numbers out of $v = 90$ and every number has the same probability of being drawn, then the probability of being drawn is obviously:

$p_j = p = \frac{n}{v}$ for all the v numbers.

If the draw is repeated N times, the frequencies k_j of the numbers j occurring among the numbers drawn can be determined from the data of the N drawings, similarly to roulette-type games. However, here for the frequencies k_j obviously the following conditions have to be fulfilled:

$$k_j = 0, 1, 2, \ldots, N \text{ and } \sum_{j=1}^{\nu} k_j = Nn.$$

Every frequency k_j follows the same Bernoulli distribution:

$$P(k_j) = \binom{N}{k_j} p_j^{k_j} (1-p_j)^{N-k_j} = \binom{N}{k_j} p^{k_j} (1-p)^{N-k_j}$$

with the expectation $\langle k_j \rangle \equiv \mu = Np$ and with the variance $\langle (k_j-\mu)^2 \rangle \equiv \sigma^2 = Np(1-p)$.

It is quite logical on the basis of the foregoing to define the next random variable:

$$\chi_L^2 = \sum_{j=1}^{\nu} \frac{(k_j - Np)^2}{Np} \tag{4}$$

similarly to the case of roulette, and to hope that its distribution extends to a known distribution limit, for example to the evident χ^2 distribution, if $N \to \infty$. The distribution limit of this variable χ_L^2 (where L refers to the lottery) cannot be derived on the basis of the theorem shown in paragraph 3 above, since the conditions of use of that theorem are not fulfilled here; the set of frequencies k_j *does not follow a multivariable Bernoulli distribution*, however, every k_j in itself is of (simple) Bernoulli distribution. Even in principle it is impossible that every number occurs Nn times (as in the case of roulette), though the number of all the drawn numbers equals Nn. Every number j can occur N times at most. However, it is a lucky circumstance, that for $n = 1$ the variable χ_L^2 is the same as the variable χ_R^2, if the relation $p_j = 1/\nu$ is true. Therefore their distributions (and consequently their distribution limits) must be the same as well. This fact makes it easier to check the correctness of the relations obtained.

In order to "guess" the distribution limit required, let us determine the expectation of the variable χ_L^2.

$$\langle \chi_L^2 \rangle = \sum_{j=1}^{\nu} \frac{\langle (k_j - Np)^2 \rangle}{Np} =$$

$$= \sum_{j=1}^{\nu} \frac{\sigma^2}{Np} = \frac{\nu\sigma^2}{Np} = \frac{\nu Np(1-p)}{Np} = \nu(1-p) = \nu - n$$

Is the distribution limit that is sought after a χ^2 distribution with a degree of freedom $\nu - n$? Let us also determine the variance of χ_L^2. As a result of calculations that are more complicated than the foregoing,

$$\sigma_{\chi_L^2}^2 \equiv \left\langle \left(\chi_L^2 - \langle \chi_L^2 \rangle \right)^2 \right\rangle = 2\frac{(\nu-n)^2}{\nu-1}\left(1 - \frac{1}{N}\right) \text{ can be given.}$$

Because for $N \to \infty$ $\quad \sigma_{\chi_L^2}^2 \to 2\frac{(\nu-n)^2}{\nu-1}$, it can be guessed that the distribution of the next variable Y

$$Y \equiv \frac{\nu-1}{\nu-n}\chi_L^2 \equiv \frac{\nu-1}{\nu}\sum_{j=1}^{\nu}\frac{(k_j-Np)^2}{Np(1-p)} \tag{5}$$

goes to a χ^2 distribution with $\nu - 1$ degrees of freedom, since the expectation of the variable Y equals $\mu_Y = \nu - 1$ and its variance is:

$$\sigma_Y^2 = 2(\nu-1)\left(1-\frac{1}{N}\right) \to 2(\nu-1)$$

Indeed, using the method of the characteristic functions it was proved that for the case of $N \to \infty$

$$\phi_Y = F_{\nu-1}(x) \tag{6}$$

The theorem can also be proved for the more general case, when the probabilities p_j belonging to the event, that the number j is occurring among the n numbers drawn, are not equal, but this proof is quite complex and is not necessary for our purposes.

Applying the above theorem the χ^2 test can also be used for lottery-type games, if the number N of the draw results, in the list available for the tests, is great enough.

6 Improvement of the χ^2 test for lottery-type games

Sometimes huge prizes are drawn in lotteries, which is why it is of special importance that the mathematical test methods of the randomness of the games must be indisputable. For this purpose the second and third members of the distribution function of the variable Y, given by formula (5), were defined as well. Hence the distribution function:

$$\phi_Y(x) = F_{\nu-1}(x) -$$

$$- \frac{x}{6N}\frac{f_{\nu-1}(x)}{\nu+1}\left[3(\nu+1-x) + \varepsilon\left(\nu+1-2x+\frac{x^2}{\nu+3}\right)\right] +$$

$$+ O\left(\frac{1}{N^{3/2}}\right)$$

where:

$$\varepsilon \equiv \frac{(v-1)(v-2n)^2}{n(v-2)(v-n)}$$

It can be seen that n appears only in the formula of ε. The values of ε for $n = 1$ and for $n = v - 1$ are the same, therefore the distributions of Y must also be the same in these two cases. This fact is not unexpected at all, since the task of choosing one number out of v is equivalent to the task of choosing $v - 1$ different numbers out of v. At the same time, for $n = 1$ the distribution function of the variable Y is equal to the distribution function of the variable χ^2_R, if

$$p_j = \frac{1}{v}, \text{ i.e. } \sum_{j=1}^{v} \frac{1}{p_j} = \sum_{j=1}^{v} v = v^2 \qquad (7)$$

is substituted therein, as it must be, since fulfilling condition (7) - i.e. in the case of uniformly distributed probabilities p_j - the variable χ^2_R forms a special case of the variable Y.

Because in the case of lottery-type games the order in which the numbers are drawn is not of any significance, here it is not reasonable to perform the χ^2 test for the modulated differences $z_{i,k}$ too. The sequence of the results of lottery drawings has not been recorded in statistics for several decades; the numbers drawn are reported only in increasing order.

7 Conclusions

One means (and in the given cases the only possible means) of testing gambling devices is the mathematical statistical checking of draw or spin results. This paper shows the 3σ and the χ^2 tests, which are most frequently used. The advantages and disadvantages of both methods are given, on the basis of which it can be established that the two methods are complementary to each other, therefore if possible both should be used. If a χ^2 test results in a rejection, it is useful to perform the 3σ test too in order to ascertain the reasons for this rejection.

The conventional χ^2 method cannot be used for testing lottery-type games, which is why the author has put forward additional calculations that allow the χ^2 test to be used for this purpose as well.

Considering that the χ^2 test is used not "only" for checking scientific hypotheses but also for establishing an official decision about acceptance or rejection, the improvement to the χ^2 test was made for both kinds of games, i.e. for roulette and lottery-type games. This gives the possibility to *prove* the correctness of use of the first approximation or to use the χ^2 test in cases when the conditions for using the first approximation cannot be fulfilled, for instance when the number N of the experiments is not big enough, or when the probability distribution of the events A_j is not uniform. ■