UNCERTAINTY OF MEASUREMENT

Method for calculating the coverage factor in calibration

PAWEL FOTOWICZ, General Metrology Division
Central Office of Measures, Poland

1 Introduction

In accordance with the provisions of the European Co-operation for Accreditation, the expanded uncertainty of measurement in calibration should be expressed for a coverage probability of approximately 95 % [1]. In cases where a normal distribution can be attributed to the measurand and the standard uncertainty associated with the output estimate has sufficient reliability, the standard coverage factor $k = 2$ shall be used. The assumption of normal distribution cannot always be easily confirmed. The standard coverage factor $k = 2$ can yield an expanded uncertainty corresponding to a coverage probability different to 95 %. The use of approximately the same coverage probability is essential whenever two results of measurement of the same quantity have to be compared, e.g. when evaluating the results of an inter-laboratory comparison or assessing compliance with a specification. In these cases, in order to ensure that the value of the expanded uncertainty is quoted corresponding to the same coverage probability as in the normal case, another method has to be employed.

In the reference publication EA-4/02 (supplement 2), two approximation methods for coverage factor calculation are proposed. One method relies on approximation of the output quantity distribution by a rectangular distribution in cases where a dominant contribution in the budget is a quantity having a rectangular distribution. In this situation the coverage factor is $k = 1.65$ for a coverage probability of 95 %. The second method relies on approximation of the output quantity distribution by a trapezoidal distribution in cases where dominant contributions in the budget are two quantities having rectangular distributions. In this situation the coverage factors are from $k = 1.65$ to $k = 1.9$ for a coverage probability of 95 %. The value of the coverage factor depends on the ratio of the uncertainty of the dominant contributions and is given by:

$$k = \sqrt{\frac{3(r+1)^2}{r^2+1}}\left(1 - \sqrt{(1-p)\frac{4r}{(r+1)^2}}\right) \quad \text{for } 1 \le r \le 10 \qquad (1)$$

where:

p = coverage probability

$r = \left|\frac{u_1(y)}{u_2(y)}\right|$ = ratio of the dominant contributions

$u_1(y)$ i $u_2(y)$ = dominant uncertainty contributions

These methods do not solve the problem in the general case where there are several contributions in the budget with normal and rectangular distributions having various standard uncertainties, but non-dominant terms. In this situation the conditions of the Central Limit Theorem are not met and it cannot be assumed that the distribution of the output quantity is normal. The output quantity distribution is the convolution of rectangular and normal distributions (R*N distribution).

2 Approximation of the convolution of rectangular and normal distributions by a symmetrical trapezoidal distribution

The coverage factor for a trapezoidal distribution given by (1) can be presented as a function of ratio r, as in Fig.1. It can be illustrated by the curve k_T. If r is the ratio of standard deviations of convolved rectangular and normal distributions, it may show the curve k_{RN} for the distribution resulting from the R*N convolution on the basis of the coverage factor value presented in Reference [3]. The difference between values of coverage factors k for the trapezoidal distribution and for the R*N distribution are small; its variability is presented in Fig. 2. The curve in this Figure shows the difference of coverage factors k_T and k_{RN} given by:

$$\delta_T = \frac{k_T - k_{RN}}{k_{RN}} \cdot 100\% \qquad (2)$$

The deviation δ_T does not exceed ± 1.5 % for a coverage probability of 95 %. This approximation can be compared with the one resulting from the traditional standpoint, in other words with the approximation of a coverage factor for R*N distribution by a coverage factor for normal distribution or for rectangular distribution. This situation is presented in Fig. 3, which represents the functions of coverage factor deviations for R*N distribution approximated by normal distribution (δ_N), by rectangular distribution (δ_R) and by trapezoidal distribution (δ_T). The suitable functions are formulated by:

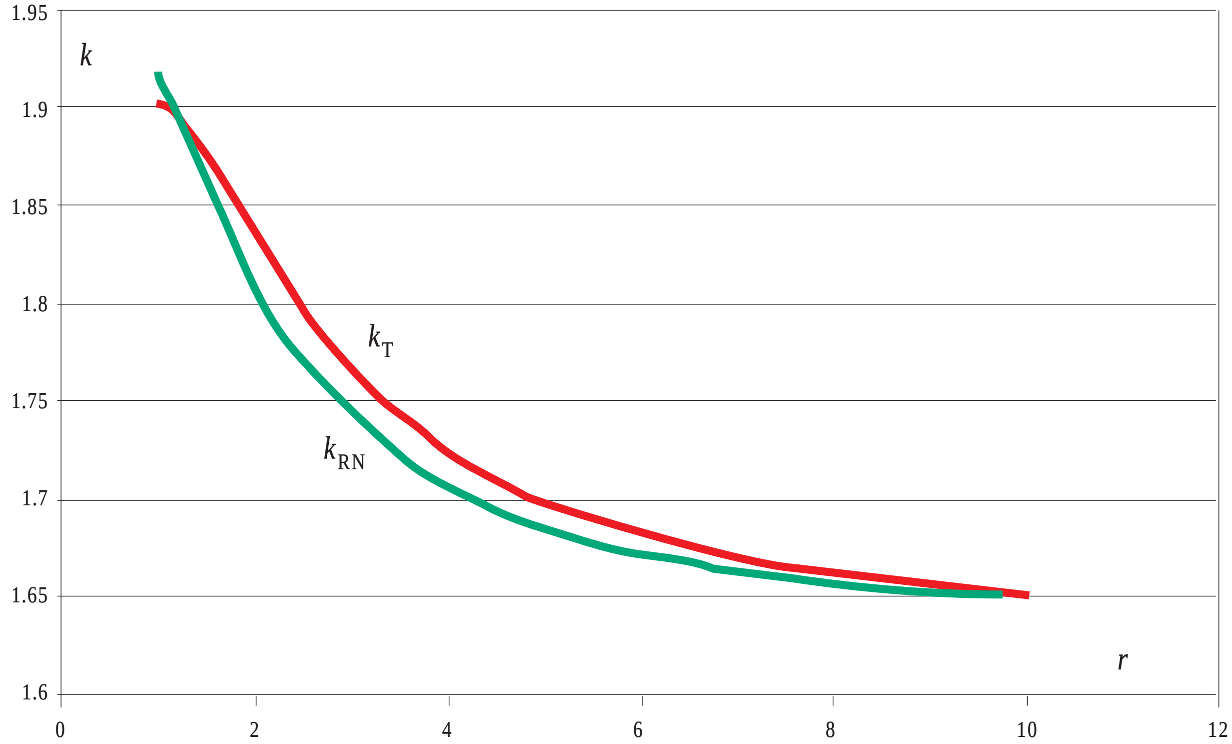

Fig. 1 Coverage factor functions for a coverage probability of 95 %

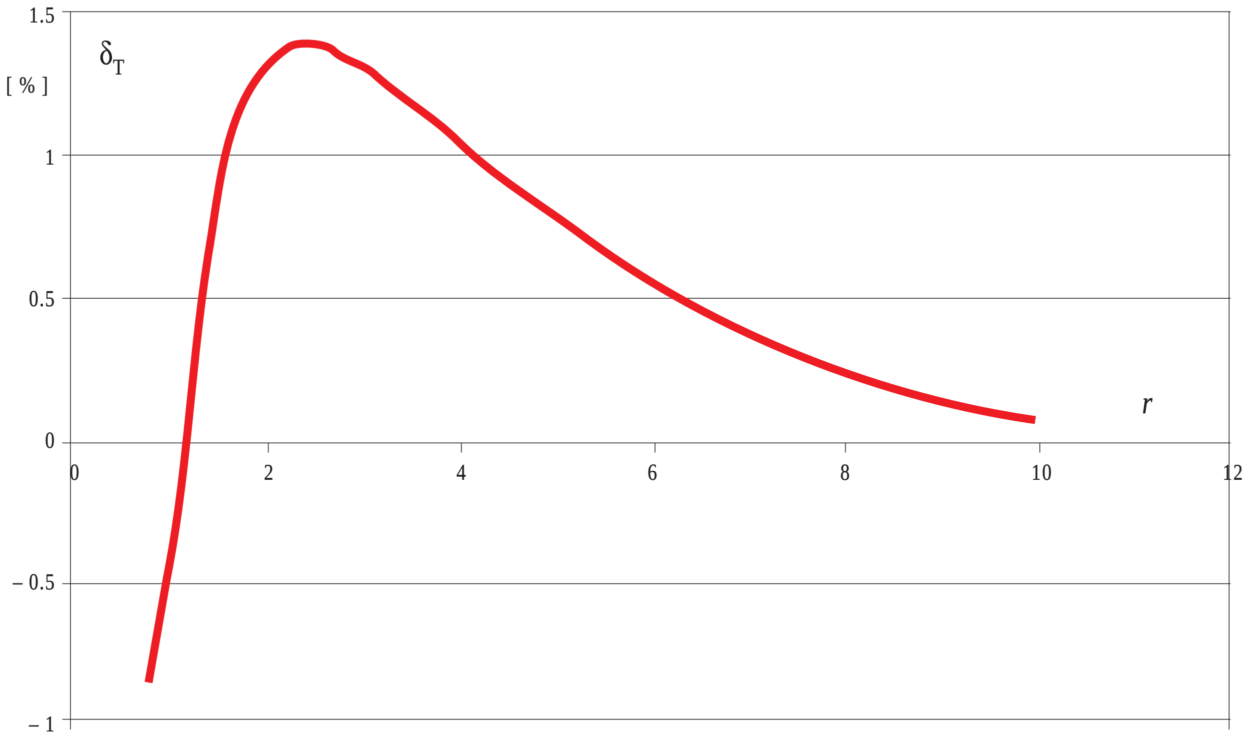

Fig. 2 Function of deviation δ_T for a coverage probability of 95 %

$$\delta_N = \frac{k_N - k_{RN}}{k_{RN}} \cdot 100\% \qquad (3)$$

$$\delta_R = \frac{k_R - k_{RN}}{k_{RN}} \cdot 100\% \qquad (4)$$

where:

k_N = the coverage factor for a normal distribution

k_R = the coverage factor for a rectangular distribution

The above consideration implies the conclusion that the best accurate approximation of the R*N distribution is a trapezoidal distribution in the range from $r = 1$ to $r = 10$. It should be expected in other ranges of r that the good approximation of the R*N distribution is a normal distribution for $r < 1$ and a rectangular distribution for $r > 10$.

3 Principle of approximation of the distribution and of the coverage factor for output quantity in calibration

On the basis of Reference [3] the coverage factor given by the R*N distribution for a coverage probability of 95 % can change in the range from $k = 1.65$ to $k = 2$ (exactly $k = 1.96$). Those extreme values correspond to the coverage factor given by the rectangular distribution and by the normal distribution. The other value of the coverage factor corresponds to the "intermediary" distribution between the rectangular and normal distributions.

Those "intermediary" distributions are the convolutions of rectangular and normal distributions (R*N distributions) with a different parameter r (ratio of the standard deviation of the rectangular distribution to the standard deviation of the normal distribution). From Fig. 1 it can be noted that the curve k_{RN} is close to the curve k_T in the range of r from 1 to 10. The trapezoidal distribution is the convolution of two rectangular distributions. This implies the conclusion that each convolution of the rectangular and "intermediary" distributions gives a k function close to the k_{RN} and k_T functions. Therefore, using an approximation of the convolution of the rectangular and "intermediary" distributions by trapezoidal distribution involves the error of approximation of the coverage factor for this convolution not larger than for the cases where the coverage factor k_{RN} is approximated by the coverage factor k_T. The "intermediary" distribution may be the convolution of several rectangular and normal distributions. For instance the coverage factors at the coverage probability of 95 %:

- given by the convolution of three identical rectangular distributions is $k = 1.94$ [2],
- given by the convolution of two identical rectangular distributions is $k = 1.90$, and
- given by the convolution of the normal and rectangular distributions with equal standard deviations is $k = 1.92$ [3].

On the basis of this analysis the following principle of approximation of output quantity distribution can be formulated: for the output quantity $Y = c_1X_1 + \ldots + c_NX_N$, where all input quantities $X_1, \ldots , X_N$ are independent and the quantity X_i having a rectangular distribution with the largest contribution $u_i(y) = c_i u(x_i)$ satisfies the condition:

$$|u_i(y)| \geq \sqrt{u_c^{\,2}(y) - u_i^{\,2}(y)}$$

the best approximation of the output quantity distribution is a trapezoidal distribution or a rectangular distribution, independently from the distributions of other input quantities. In other cases the best approximation of output quantity distribution is a normal distribution.

The following coverage factor formulae can be deduced:

$k = k_N$ for $0 \leq r < 1$

$k = k_T$ for $1 \leq r \leq 10$

$k = k_R$ for $r > 10$

where:

$$r = \frac{|u_i(y)|}{\sqrt{u_c^{\,2}(y) - u_i^{\,2}(y)}} \qquad (5)$$

$u_i(y)$ = the largest contribution of the input quantity having a rectangular distribution

k_N = coverage factor for a normal distribution

k_T = coverage factor for a trapezoidal distribution

k_R = coverage factor for a rectangular distribution

$$k_T = \sqrt{\frac{3(r+1)^2}{r^2+1}}\left(1 - \sqrt{(1-p)\frac{4r}{(r+1)^2}}\right) \qquad (6)$$

$$k_R = \sqrt{3}\,p \qquad (7)$$

p = coverage probability

4 Summary

The method presented for approximating the coverage factor of the convolution of rectangular and normal distributions has been applied. The function of an

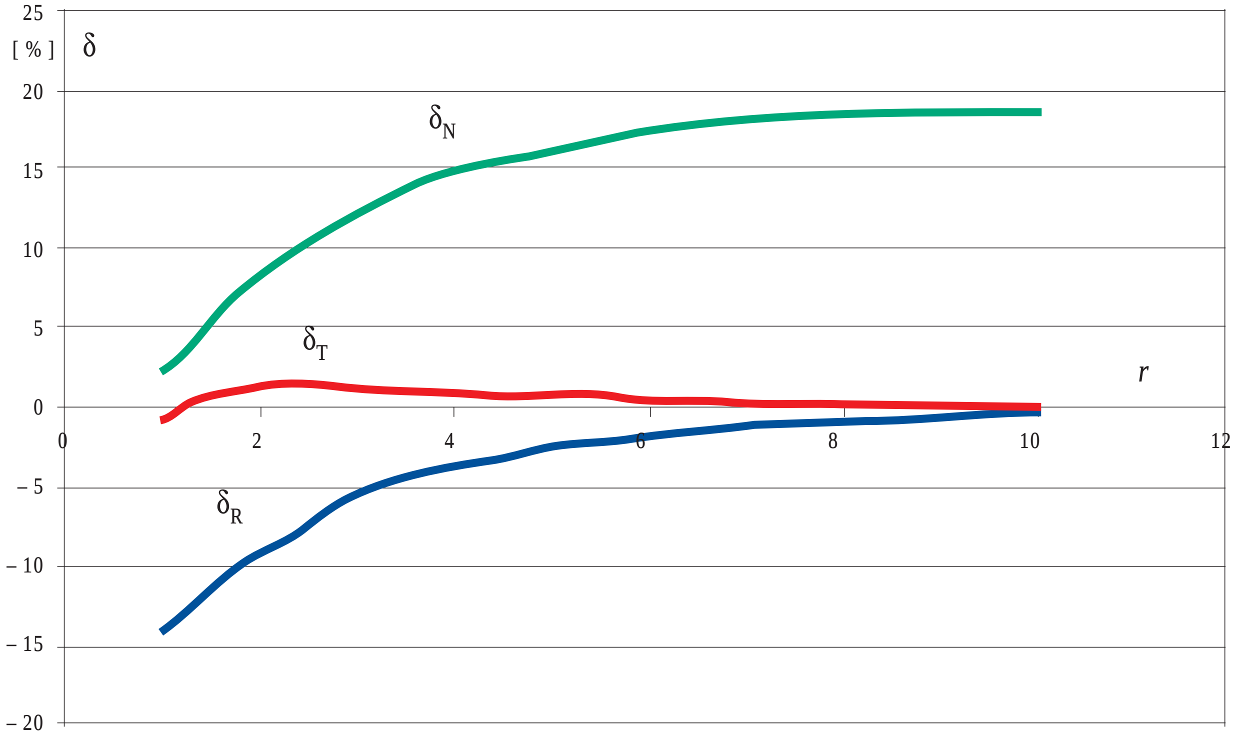

Fig. 3 Functions of deviations δ_N, δ_T and δ_R for a coverage probability of 95 %

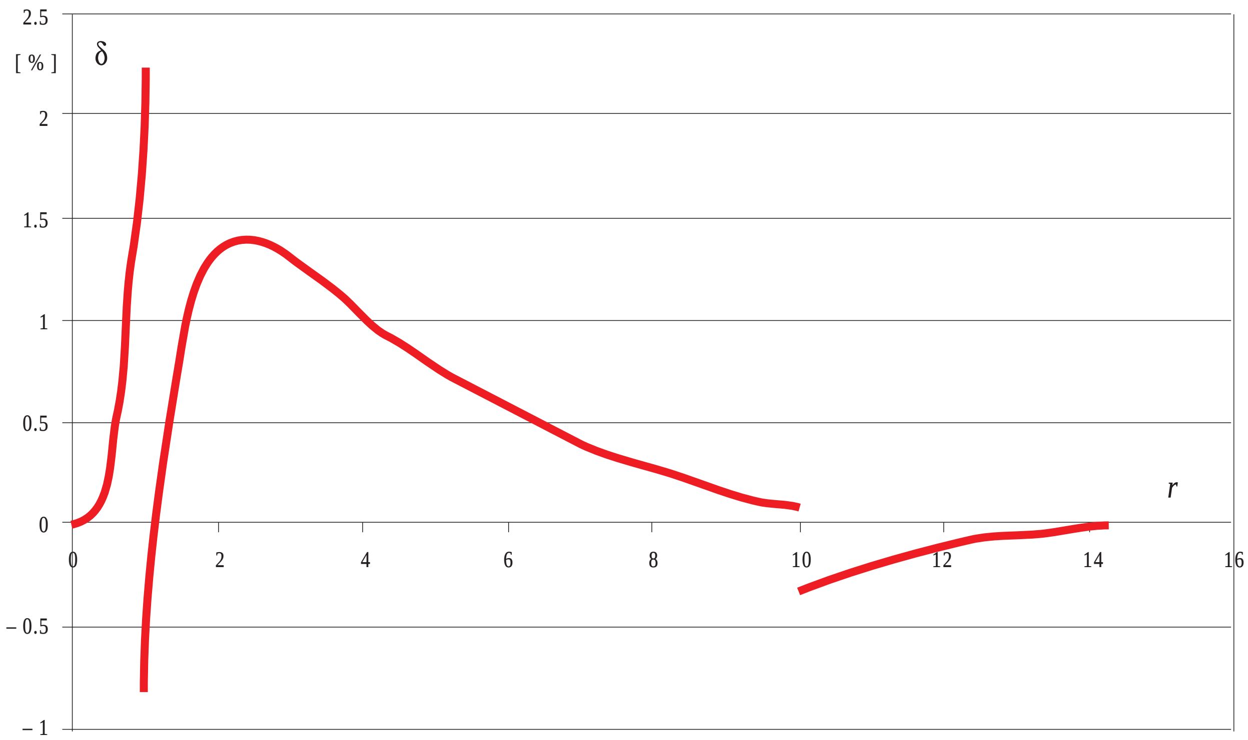

Fig. 4 Function of the error of the approximation method of the coverage factor in calibration

approximation error of the method defined by the formula is presented in Fig. 4:

$$\delta = \frac{k - k_{RN}}{k_{RN}} \cdot 100\% \tag{8}$$

The maximum value of the error for the full range of r lies in the range ± 2 %. Therefore this value of error is not larger than the value of error that occurs when the approximation is made using the normal distribution and consequently coverage factor $k = 2$ at a coverage probability of 95 %.

The method presented for approximating the output quantity distribution rendered possible an accurate estimation of the coverage factor for many input quantities having normal, rectangular and "intermediary" distributions. The method presented ensures a coverage probability of approximately 95 % for expanded uncertainty evaluation in calibration. The method may be applied in procedures for calculating the uncertainty of measurement in calibration. ■

References

[1] Expression of the Uncertainty of Measurement in Calibration. Publication Reference EA-4/02, December 1999

[2] Guide to the expression of uncertainty in measurement (GUM), BIPM, IEC, IFCC, ISO, IUPAC, IUPAP, OIML, Corrected & Reprinted Edition, 1995

[3] C.F. Dietrich: Uncertainty, Calibration and Probability. The statistics of Scientific and Industrial measurement. Second edition. Adam Hilger. Bristol, Philadelphia and New York, 1991